$T : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ linear change of coordinates

$\Rightarrow \quad T : \mathbb{P}^n \to \mathbb{P}^n$ projective change of coordinates

Fact: 1) $V \subset \mathbb{P}^n$ alg. set $\Rightarrow V^T := T^{-1}(V) \subset \mathbb{P}^n$ alg. set.

Pf: $V^T = V(F_1^T, \cdots, F_r^T)$ (where $F_i^T = F_i(T_1, \cdots, T_{n+1})$)

2). $V = $ var. $\Leftrightarrow V^T = $ var.

3). $\tilde{T} : \Gamma_h(V) \to \Gamma_h(V^T)$

$\tilde{T} : k(V) \to k(V^T)$

$\tilde{T} : \mathcal{O}_P(V) \to \mathcal{O}_{T^{-1}(P)}(V^T)$

4.3. affine and projective varieties.

$\varphi_{n+1} : \mathbb{A}^n \xrightarrow{\cong} U_{n+1} \subseteq \mathbb{P}^n$

aim: alg. sets in $\mathbb{A}^n$ & alg sets in $\mathbb{P}^n$.

4.3.1 $\forall V =$ alg. set in $\mathbb{A}^n$. $I = I(V) \triangleleft k[x_1, \cdots, x_n]$

4.20
$I = (F_1 \cdots F_r) \not\Rightarrow I^* = (F_1^*, \cdots, F_r^*)$

$I^* \triangleleft k[x_1, \cdots, x_{n+1}]$ ideal generated by F^* for all $F \in I$.

$V^* := V(I^*) \subseteq \mathbb{P}^n$. ⟵ projective closure of V

直观几何解释

Prop 3: $V, W \subseteq \mathbb{A}^n$ alg. sets Then

(1). $V^* =$ smallest alg. set in $\mathbb{P}^n$ containing $\varphi_{n+1}(V)$

(2) $\varphi_{n+1}(V) = V^* \cap U_{n+1}$

(3) $V \subseteq W \Rightarrow V^* \subseteq W^*$

(4) V irr. $\Rightarrow V^*$ irr

(5). $V = \bigcup V_i$ irr. decomp. $\Rightarrow V^* = \bigcup V_i^*$ irr. decomp.

(6) $V = \bigcup V_i \subsetneqq \mathbb{A}^n$ $(\neq \phi)$ $\Rightarrow V_i^* \not\subseteq H_\infty$ & $V_i^* \not\supseteq H_\infty$.

Pf: (2) $V^* \cap U_{n+1} := \{ [x_1 : \cdots : x_n : 1] \mid F^*(x_1, \cdots, x_n, 1) = 0 \ \forall F \in I \}$

$= \{ \varphi_{n+1}((x_1, \cdots, x_n)) \mid F(x_1, \cdots, x_n) = 0 \ \forall F \in I \}$

$= \varphi_{n+1}(V)$.

(3) $V \subseteq W \Rightarrow I(V) \supseteq I(W) \Rightarrow I(V)^* \supseteq I(W)^*$

$\Rightarrow V(I(V)^*) \subseteq V(I(W)^*) \Rightarrow V^* \subseteq W^*$

(4) $V = \text{irr} \Rightarrow I(V) = \text{prime} \Rightarrow I(V)^* = \text{prime}$

$\Rightarrow V^* = \text{irr}.$

(1). $Z \subseteq \mathbb{P}^n$ alg. set containing $\varphi_{n+1}(V)$

$F \in I(Z) \Rightarrow F_* \in I(V)$

$\Rightarrow F = X_{n+1}^r (F_*)^* \in I(V)^*$

$\Rightarrow I(Z) \subseteq I(V)^* \Rightarrow Z \supseteq V^* \Rightarrow \checkmark$

(5) $\Leftarrow$ (1), (2), (3), (4)

(6) assume $V = \text{irr}$.

- (2) $\Rightarrow V^* \nsubseteq H_\infty$
- Suppose $V^* \supseteq H_\infty$.

$\Rightarrow I(V)^* \subseteq I(V^*) \subseteq I(H_\infty) = (X_{n+1})$

$\forall F \in I(V) \setminus \{0\} \Rightarrow F^* \in I(V)^*$ & $F^* \notin (X_{n+1})$ ↯

$\Rightarrow V^* \not\supseteq H_\infty.$

4.3.2 • $\forall V =$ alg. set in $\mathbb{P}^n$. $I = I(V) \triangleleft k[X_1, \cdots, X_{n+1}]$

$I_* \triangleleft k[X_1, \cdots, X_n]$ ideal generated by F_* for all $F \in I$.

$V_* := V(I_*) \subseteq \mathbb{A}^n$.

Prop 3': $V, W \subseteq \mathbb{P}^n$ alg. sets. Then

(1) $V \subseteq W \Rightarrow V_* \subseteq W_*$

(2). if $V = \cup V_i \subseteq \mathbb{P}^n$ with $V_i \not\subseteq H_\infty$ & $H_\infty \not\subseteq V_i$ $(\forall i)$. then
$V_* \subsetneq \mathbb{A}^n$ & $(V_*)^* = V$.

(3). $V(\neq \phi) \subseteq \mathbb{A}^n$ alg set. $\Rightarrow (V^*)_* = V$

Pf: (1) $V \subseteq W \Rightarrow I(V) \supseteq I(W) \Rightarrow I(V)_* \supseteq I(W)_*$

$\Rightarrow V_* \subseteq W_*$

(2). WMA: $V = \text{irr}$.

ONTS: $I(V_*)^* \subseteq I(V)$ $\left(\Leftrightarrow V \subseteq (V_*)^* \Leftarrow \varphi_{n+1}(V_*) \subset V\right)$

$\forall F \in I(V_*) \Rightarrow F^N \in I(V)_*$ for some N

$\Rightarrow X_{n+1}^t (F^N)^* \in I(V)$ for some t

$\left.\begin{array}{l} V = \text{irr} \Rightarrow I(V) = \text{prime} \\ V \not\subseteq H_\infty \Rightarrow X_{n+1} \notin I(V) \end{array}\right\} \Rightarrow F^* \in I(V) \Rightarrow V$

(3). clear